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LETTER TO THE EDITOR

Stokes' phenomenon at second-order pole and reflectionless potentials

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Abstract. Some classes of analytic potentials with second-order poles and corresponding asymptotic solutions of the Schrödinger equation are studied. The main goal of the analysis is to answer the question as to whether a Stokes phenomenon and therefore a reflection takes place for those equations. It is shown that one can notice the Stokes phenomenon not only in the leading term of an asymptotic expansion but also in higher-order ones. Therefore for only a restricted set of potentials proposed, the reflection coefficient is exactly equal to zero although it looks like that for other potentials in the first approximation. This letter gives rise to a further extension of the original approach of Berry and Howls.

In a paper by Berry and Howls (1990) a very fruitful approach to the reflectionlessness phenomenon for the one-dimensional Schrödinger equation was formulated which is completely different from the original one (Kay and Moses 1956). Their main idea is that reflectionlessness takes place if the asymptotics at transition points in a complex plane is constructed not with the help of Airy or Weber functions but with the help of 'fake' functions which in fact can be expressed in terms of Bessel functions of half-integer order. The Stokes phenomenon for the latter functions does not take place since they are elementary. Therefore there exist solutions for which asymptotics consist only of one exponent in the vicinity of the transition point. These solutions correspond to waves without reflections. That is different from an ordinary case where the reflection coefficient is exponentially small but still not zero (Pokrovskii *et al* 1958). The significance of reflectionless potentials in soliton theory is well known. Therefore any new possibility to introduce a class of reflectionless profiles is worthy of further considerations.

In our letter we try to propose further details for Berry's formalism. The mathematical treatment is mostly formal although we are convinced that strict proofs can be given as well.

We shall begin with a model equation of the form

$$y''+[p^2q(x)+r(x)-\mu/x^2]y(x)=0 \tag{1}$$

where p is a large parameter ($p \gg 1$) and try to answer whether a Stokes phenomenon takes place for the solution of equation (1) at the point $x=0$.

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Functions $q(x)$ and $r(x)$ are considered to be analytical functions in a vicinity of the real axis. Different possible cases of the shape of $q(x)$ are considered.

(a) $q(x)>0$ at the real axis.

(b) $q(0)=0$ $q'(0)\neq 0$ $q(x)<0$ for $x<0$

$q(x)>0$ for $x>0$.

(c) $q(x)=0$ $q(x)>0$ for $x>0$ and for $x<0$.

(d) $q(x)=q_1(x)/x$ $q_1(0)>0$.

The answer to the question as to whether the Stokes phenomenon takes place for equation (1) at $x=0$ depends upon whether a comparison equation in cases (a)-(d) can be chosen in the form:

(a) $u''+[p^2-l(l+1)/\xi^2]u=0$

(b) $u''+[p^2\xi-(6l+5)(6l+1)/(16\xi)]u=0$

(c) $u''+[p^2\xi^2-(4l+1)(4l+3)/(4\xi^2)]u=0$

(d) $u''+[p^2/\xi-(2l+3)(2l-1))/(16\xi^2)]u=0.$

Their solutions are

(a) $u=\xi^{1/2}Z_{l+1/2}(p\xi)$

(b) $u=\xi^{1/2}Z_{l+1/2}(\frac{2}{3}p\xi^{3/2})$

(c) $u=\xi^{1/2}Z_{l+1/2}(\frac{1}{2}p\xi^2)$

(d) $u=\xi^{1/2}Z_{l+1/2}(2p\xi^{1/2})$

where $Z_{l+1/2}(t)$ is any cylinder function (Bessel or Hankel) of half-integer order.

Now we have to construct a mapping $x\to\xi$ expanded in an asymptotic series of the form

$$\xi(x,p)=\sum_{k=0}^{\infty}\xi_k(x)p^{-2k} \tag{2}$$

where $\xi_k(x)$ are regular functions at the origin which transforms equation (1) in the cases (a)-(d) to one of the equations (a)-(d) respectively.

It is easy to prove that the function $\xi(x,p)$ satisfies an equation

$$\begin{aligned}&\xi'^2R_\alpha(\xi)-q(\xi)-r(x)-p^{-2}(\mu_\alpha\xi'^2\xi^{-2}-\mu x^{-2})-\tfrac{1}{2}p^{-2}\{\xi,x\}=0\\&\{\xi,x\}=\xi'''\xi'^{-1}-\tfrac{3}{2}\xi''^2\xi'^{-2}\end{aligned} \tag{3}$$

where α refers to the cases (a)-(d), so that

$$R_a=1 \qquad R_b=\xi \qquad R_c=\xi^2 \qquad R_d=\xi^{-1}$$

$$\mu_a=l(l+1) \qquad \mu_b=(6l+5)(6l+1)/16$$

$$\mu_c=(4l+1)(4l+3)/4 \qquad \mu_d=(2l+3)(2l-1)/16.$$

By substituting the expansion (2) into equation (3) and equating the subsequent terms in powers of p^{-2} one formally gets instead of (3) the following recurrence set of equations

$$\xi_0'^2R_\alpha(\xi_0)=q(x) \tag{4}$$

$$2\xi_0'R_\alpha^{1/2}(\xi_0)\,\mathrm{d}/\mathrm{d}x[\xi_1R_\alpha^{1/2}(\xi_0)]=r(x)+\mu_\alpha\xi_0'^2\xi_0^{-2}-\mu x^{-2}+\tfrac{1}{2}\{\xi_0,x\} \tag{5}$$

$$2\xi_0'R_\alpha^{1/2}(\xi_0)\,\mathrm{d}/\mathrm{d}x[\xi_nR_\alpha^{1/2}(\xi_0)]=\mu_\alpha F_{1n}(x)+F_{2n}(x). \tag{6}$$

where $F_{1n}(x)$ and $F_{2n}(x)$ are functions of $\xi_0, \ldots, \xi_{n-1}$. It is easy to obtain that a regular solution of the equation (4) is given by the formula

$$\int_0^{\xi} R_\alpha^{1/2}(\xi)\,\mathrm{d}\xi = \int_0^x q^{1/2}(x)\,\mathrm{d}x.$$

According to this formula the function $\xi_0 \to x$ is expanded in the vicinity of the origin in a Taylor series of the form

$$\xi_0(x) = \sum_{j=1}^{\infty} a_j x^j \qquad a_1 \neq 0.$$

Now if one wants to exclude a second-order pole on the right-hand side of equation (5) it is necessary to choose the quantity μ in equation (1) equal to μ_α.

But still there exist in general a singularity of the first order. It can be smoothed up to an integrable singularity only in the case (d). In the other cases as a general fact there is no full asymptotic expansion in terms of Bessel functions of half integer order valid at the origin. Therefore more sophisticated comparison equations should be chosen. Still for potentials with additional properties this possibility exists.

For instance if in the case (a) the functions $q(x)$ and $r(x)$ are symmetrical, that is that their Taylor expansion at the origin consist of only even powers of x, then we can obtain a set of solutions of the equations (5) and (6) regular at the origin. We have to mention that an example presented in Berry's article actually belongs to this type.

The ground reason for the troubles in cases (a)-(c) are due to the fact that a second-order pole gives rise to a cluster of additional transition points at small separation of each other. If this separation is too small we do not need to treat the points of this cluster separately but if it is not small enough we have to introduce some new terms and parameters in a comparison equation which makes it more complicated and not solvable in terms of Bessel functions of half-integer order. The idea that a more simple comparison equation can be used for the first-order term and a more complicated one should be used for higher approximations was originally proposed by Olver (1975).

Now we have to extend our consideration to a larger amount of points in a complex plane with the same characteristics that we have already examined. It is evident that all of these points should have the feature that we formulated, namely that one of equations (a)-(d) can serve as a comparison equation at this point in all orders. Only in this case reflectionlessness takes place.

It is easy to construct an example of a potential where the equation (d) serves as a comparison equation that is

$$y'' + \left(\frac{p^2}{\xi^2+1} + \frac{(2l+3)(2l-1)}{4(\xi^2+1)^2}\right) y = 0.$$

This equation has solutions in terms of a hypergeometric function and there is no Stokes phenomenon for its solution. But we have to take energy to be zero so that no plane wave exists. Otherwise additional turning points exist.

We take as well an equation of the form

$$y''(x) + \left(p^2 q(x) + r(x) - l(l+1) \sum_{j=-\infty}^{\infty} \frac{1}{(x-(j+1/2)\mathrm{i}h)^2}\right) y(x) = 0$$

but in this case for the sake of symmetry at each pole the functions $q(x)$ and $r(x)$

should be periodic in a complex plane with an imaginary period ih. At the same time there should be no additional turning points.

A well known result is

$$q(x)=1$$

$$r(x)=l(l+1)\left(-\operatorname{sech}^2 z+\sum_{j=-\infty}^{\infty}\frac{1}{(z-\frac{1}{2}\mathrm{i}\pi(2j+1))^2}\right)=0.$$

There still exists the problem of proving within the presented approach that there are no other examples.

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